Embedded Systems - #2

Review on continuous processes

Maria Domenica Di Benedetto
Giordano Pola

Center of Excellence for Research DEWS
Dept of Electrical and Information Engineering
University of L'Aquila, Italy
mariadomenica.dibenedetto,giordano.pola@univaq.it

Acknowledgement

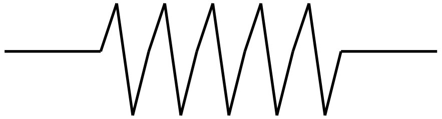
Thanks to Agung Julius for contributing his lectures at the University of Pennsylvania USA for this class

Contents

- Modeling with differential equations
- Taxonomy of systems
- Solution to linear ODEs
- General solution concept
- Simulation and numerical methods
- State space representation
- Stability

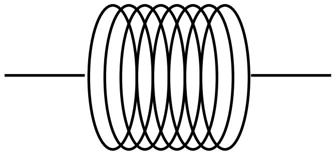
Physical systems

Resistor



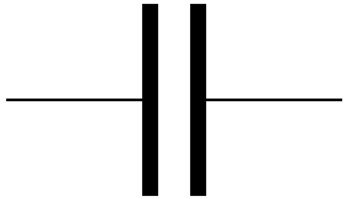
$$V(t) = R \cdot I(t)$$

Inductor



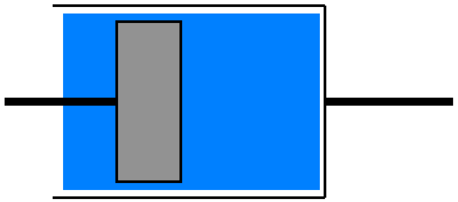
$$V(t) = L\frac{dI}{dt}$$

Capacitor



$$I(t) = C\frac{dV}{dt}$$

Damper

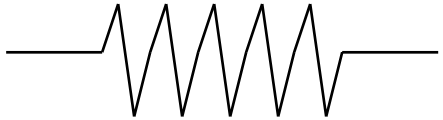


$$F(t) = b \cdot v(t)$$

Mass

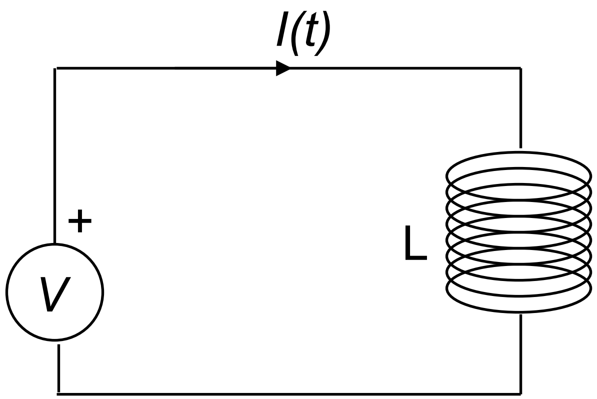
$$F(t) = M\frac{dv}{dt}$$

Spring

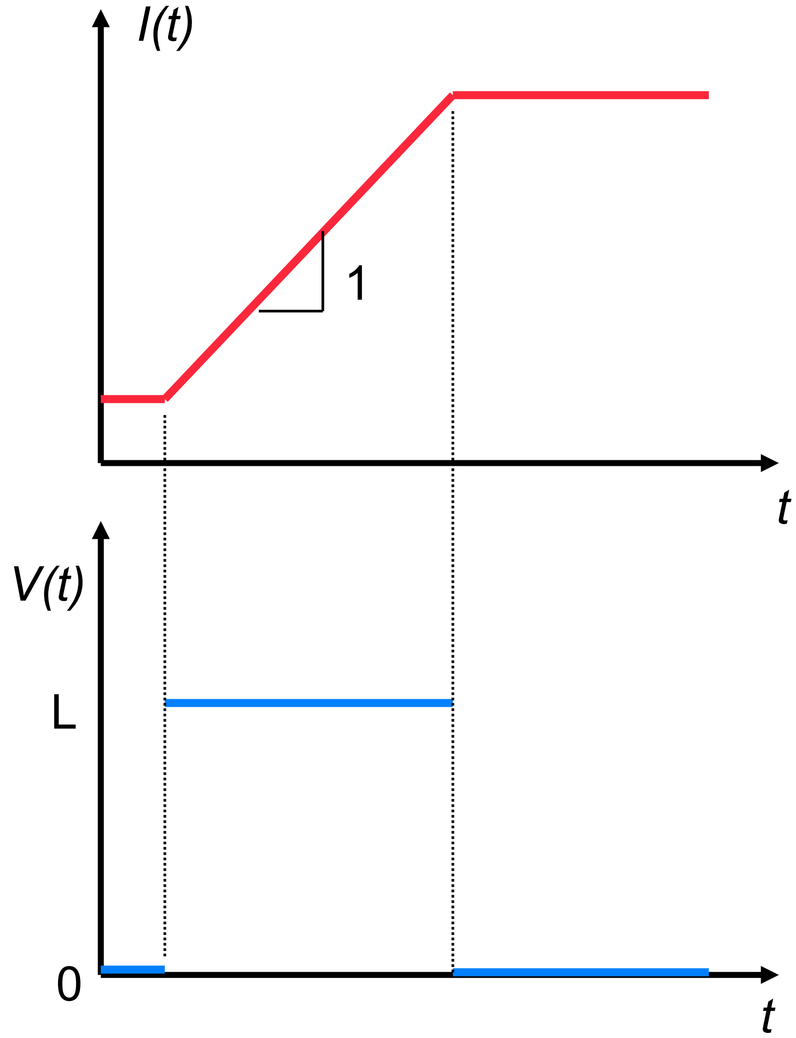


$$v(t) = \frac{1}{k}\frac{dF}{dt}$$

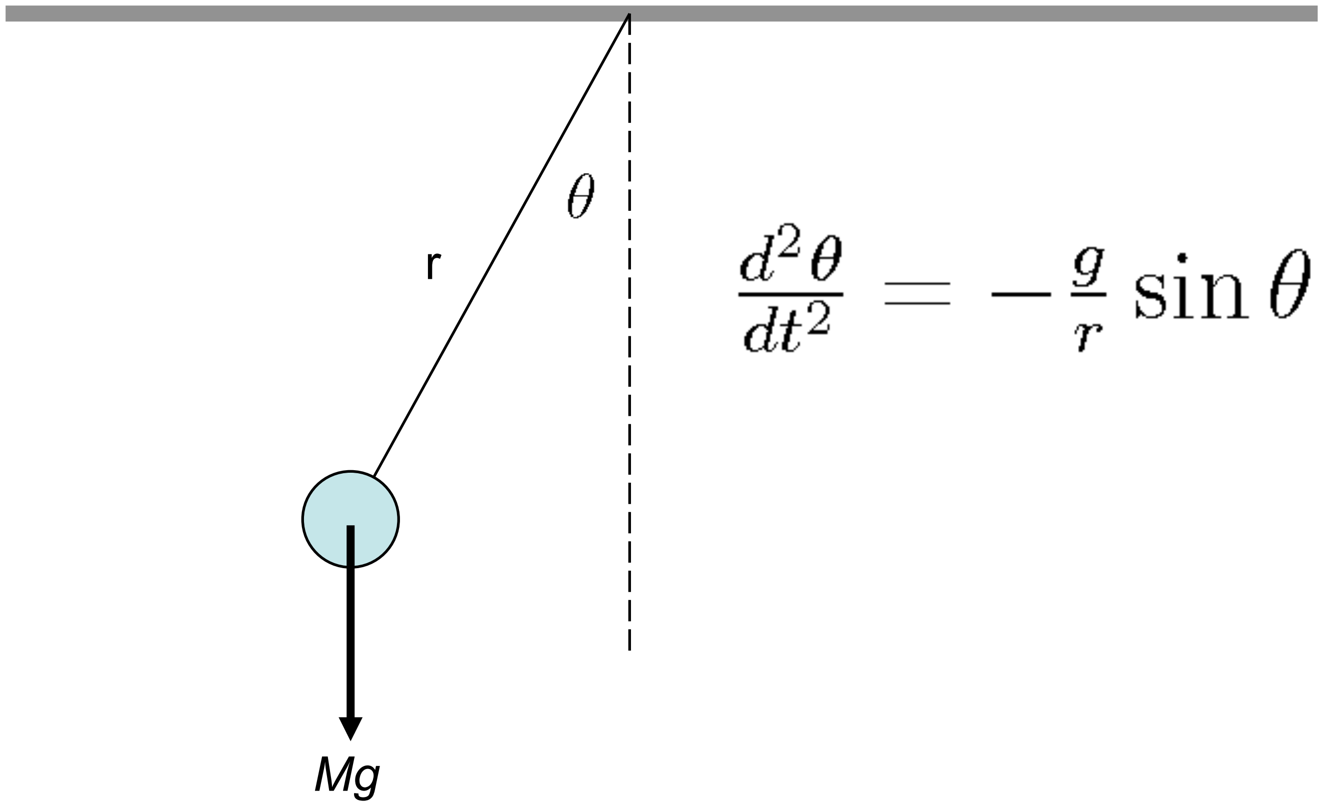
Electric circuits



$$V(t) = L\frac{dI}{dt}$$



A pendulum



$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

Contents

- Linear systems: if the set of solutions is closed under linear operation, i.e. scaling and addition

$$\left\{ \begin{array}{c} V_1(t) = L\frac{dI_1}{dt} \\ V_2(t) = L\frac{dI_2}{dt} \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} \alpha V_1(t) = L\frac{d(\alpha I_1)}{dt} \\ V_1(t) + V_2(t) = L\frac{d(I_1+I_2)}{dt} \end{array} \right\}$$

- All the examples are linear systems, except for the pendulum

$$\left\{ \frac{d^2\theta_1}{dt^2} = -\frac{g}{r}\sin\theta_1(t) \right\} \not\Rightarrow \left\{ \frac{d^2\alpha\theta_1}{dt^2} = -\frac{g}{r}\sin\alpha\theta_1(t) \right\}$$

Time invariant versus time varying

- Time invariant: the set of solutions is closed under time shifting

$$\left\{ \frac{d^2\theta_1}{dt^2} = -\frac{g}{r}\sin\theta_1(t) \right\} \Rightarrow \left\{ \frac{d^2\theta_1(t-\Delta)}{dt^2} = -\frac{g}{r}\sin\theta_1(t-\Delta) \right\}$$

- Time varying: the set of solutions is not closed under time shifting

$$\frac{dy}{dt} = tx(t)$$

Autonomous versus non-autonomous

- Autonomous systems: given the past of the signals, the future is fixed

$$\frac{d^2\theta}{dt^2} = -\frac{g}{r}\sin\theta(t)$$

- Non-autonomous systems: there is possibility for input, non-determinism

Contents

First order linear ODE:

$$\boxed{\begin{aligned}\frac{dx}{dt} &= \gamma x,\\ x(t) &= k \cdot e^{\gamma t}.\end{aligned}}$$

Higher order linear ODEs, denote the differential operator by s,

$$\{\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 2x = 0\} \Rightarrow \{s^2 + 3s + 2 = 0\}$$

Take the roots of the characteristic polynomial.

$$\boxed{x(t) = k_1 \cdot e^{-2t} + k_2 \cdot e^{-t}}$$

Use Laplace transform,

$$\mathcal{L}(x(t)) = X(s) = \int_0^\infty x(t)e^{-st}dt.$$

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0)$$

Obtain the solution in the frequency domain $X(s)$, and use inverse transform to time domain.

$$\mathcal{L}^{-1}(X(s)) = x(t) = \int_{-\infty}^{+\infty} X(s)e^{st}ds$$

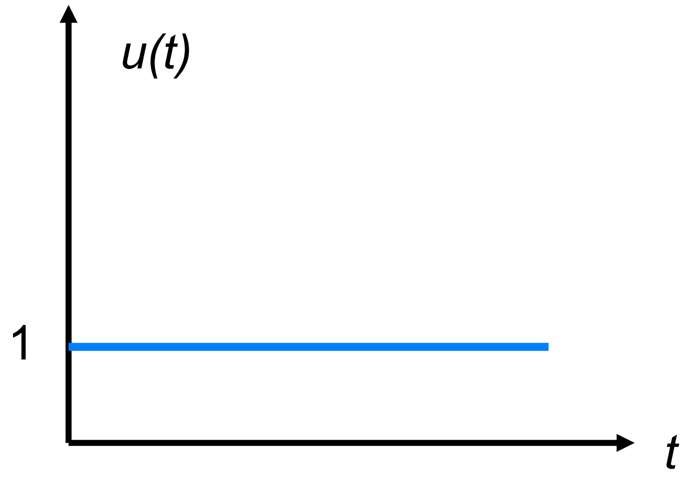
Techniques for non-autonomous systems

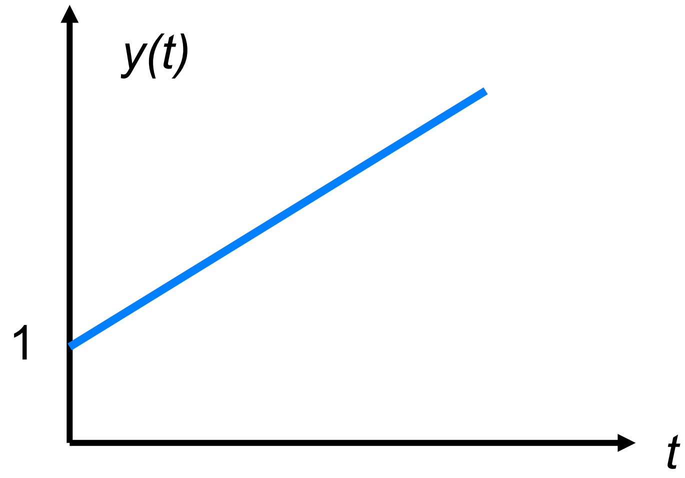
Example:

$$\frac{dy}{dt} = u(t)$$

$$u(t) = \mathbb{1}(t), y(0) = 1.$$

$$sY(s) - 1 = \frac{1}{s} \Rightarrow Y(s) = \frac{1}{s^2} + \frac{1}{s}, y(t) \quad = t\mathbb{1}(t) + \mathbb{1}(t).$$





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Solution concepts

- Given a differential equation, $\frac{dx}{dt} = f(x, u)$, and a function $\tilde{x}(t)$. When can we say that $(\tilde{x}(t), \tilde{u}(t))$ is a solution of the differential equation?
- When $\tilde{x}(t)$ is differentiable, then it is straightforward. This is called a strong solution to the equation.
- When $\tilde{x}(t)$ is not differentiable, then $(\tilde{x}(t), \tilde{u}(t)$ is a solution if there exists an x_0 such that

$$\tilde{x}(t) = x_0 + \int_0^t f(\tilde{x}(\tau), \tilde{u}(\tau))\, d\tau$$

This is called a weak solution to the equation.

Example of weak solution

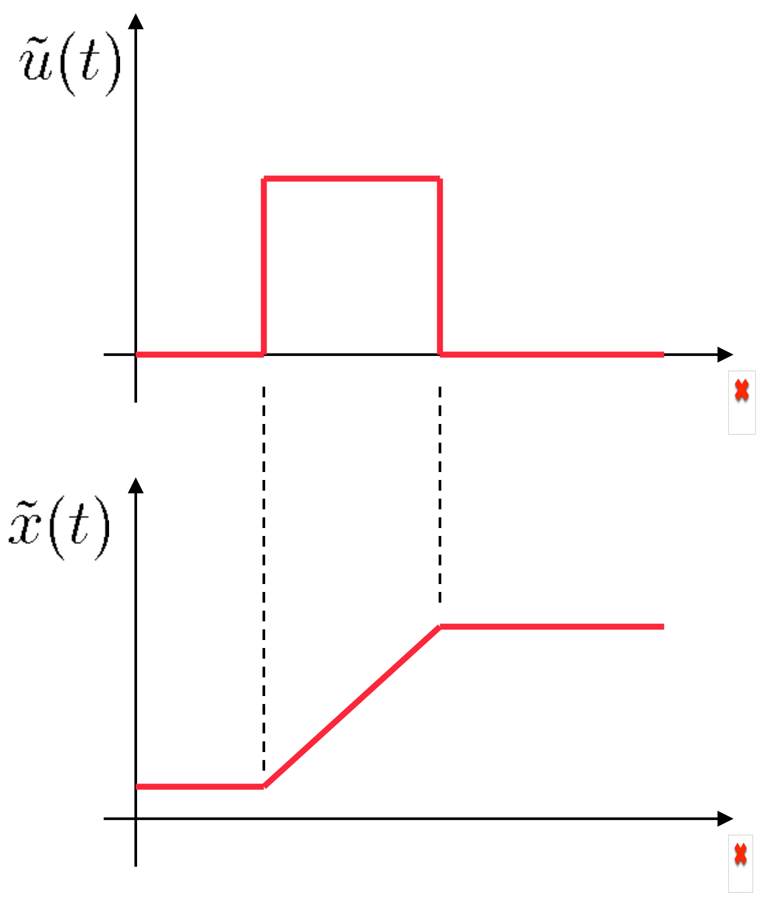
Suppose that $\frac{dx}{dt} = u(t)$.

$$\tilde{x}(t) = \begin{cases} 1/4, & t \leq 1 \\ t - 3/4, & 1 < t \leq 2 \\ 5/4, & t > 2 \end{cases},$$

$$\tilde{u}(t) = \begin{cases} 0, & t \leq 1 \\ 1, & 1 < t \leq 2 \\ 0, & t > 2 \end{cases}.$$

is a weak solution since

$\tilde{x}(t) = \frac{1}{4} + \int_0^t \tilde{u}(\tau)\, d\tau.$



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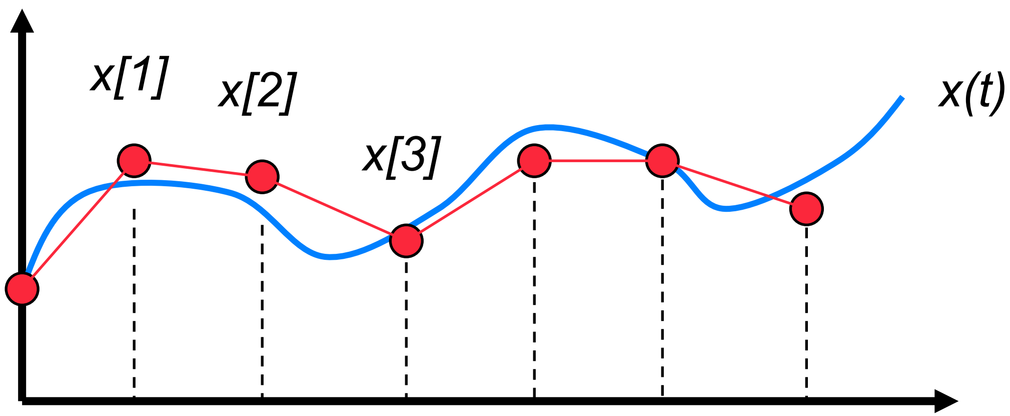
Simulation methods

- Given a differential equation $\frac{dx}{dt} = f(x,t)$.
- To simulate, i.e. numerically compute the solution, we need to discretize.



Forward difference method (Euler) : $\frac{dx}{dt} \approx \frac{x[k+1]-x[k]}{\Delta}$

$$\boxed{x[k+1] = x[k] + \Delta \cdot f(x[k], k\Delta)}$$

Simulation methods

- Backward difference method: $\frac{dx}{dt} \approx \frac{x[k]-x[k-1]}{\Delta}$

$$x[k] = x[k-1] + \Delta \cdot f(x[k], k\Delta)$$

- In each iteration we need to solve an implicit function of $x[k]$. Advantage: the algoritm is more stable.

- Exact discretization is possible for linear time invariant systems.

- There are more sophisticated algoritm, e.g. Runge-Kutta, etc. Most popular algorithms are built in features in most programming/simulation packages, such as MATLAB, MAPLE, etc.

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One of the most important representations of linear time invariant systems.

$$\frac{dx}{dt} = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t).$$

$x(t)$ is called the state of the system, $u(t)$ is the input and $y(t)$ is the output of the system. All variables are vector valued.

A, B, C, D are matrices with appropriate dimensions.

This representation is sometime also called input/state/output representation.

- Higher order input/output systems can be cast in state space representation.

$$\boxed{\ddot{y}(t) + 6\dot{y}(t) + 8y(t) = u(t),}$$
$$x_1(t) = y(t), x_2(t) = \dot{y}(t).$$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -8 & -6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

- Thus, we can transform scalar high order ODE to vector first order ODE.

Solution to state space representation

$$\boxed{\begin{aligned}\frac{dx}{dt} &= Ax(t) + Bu(t),\\ y(t) &= Cx(t) + Du(t).\end{aligned}}$$

Solution:

$$\boxed{\begin{aligned}x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau,\\ y(t) &= Ce^{At}x(0) + \int_0^t e^{CA(t-\tau)}Bu(\tau)\,d\tau + Du(t).\end{aligned}}$$

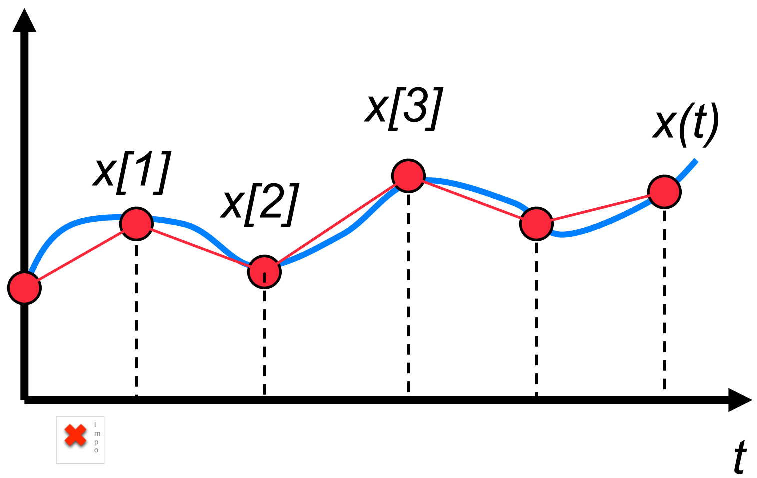
Matrix exponential: $e^A := I + A + \frac{A^2}{2!} + \cdots + \frac{A^n}{n!} + \cdots$.
Easy to compute if A is diagonal.

Alternative: $\boxed{\mathcal{L}(e^{At}) = (sI - A)^{-1}}$

Exact discretization of autonomous systems

- Consider $\dot{x} = Ax(t)$. The solution to this equation is $x(t) = e^{At}x(0)$.
- We sample the system with sampling interval Δ. We have that

$$\boxed{\begin{aligned} x(\Delta) &= e^{A\Delta}x(0), \\ x((k+1)\Delta) &= e^{A\Delta}x(k\Delta), \\ x[k+1] &= e^{A\Delta}x[k]. \end{aligned}}$$



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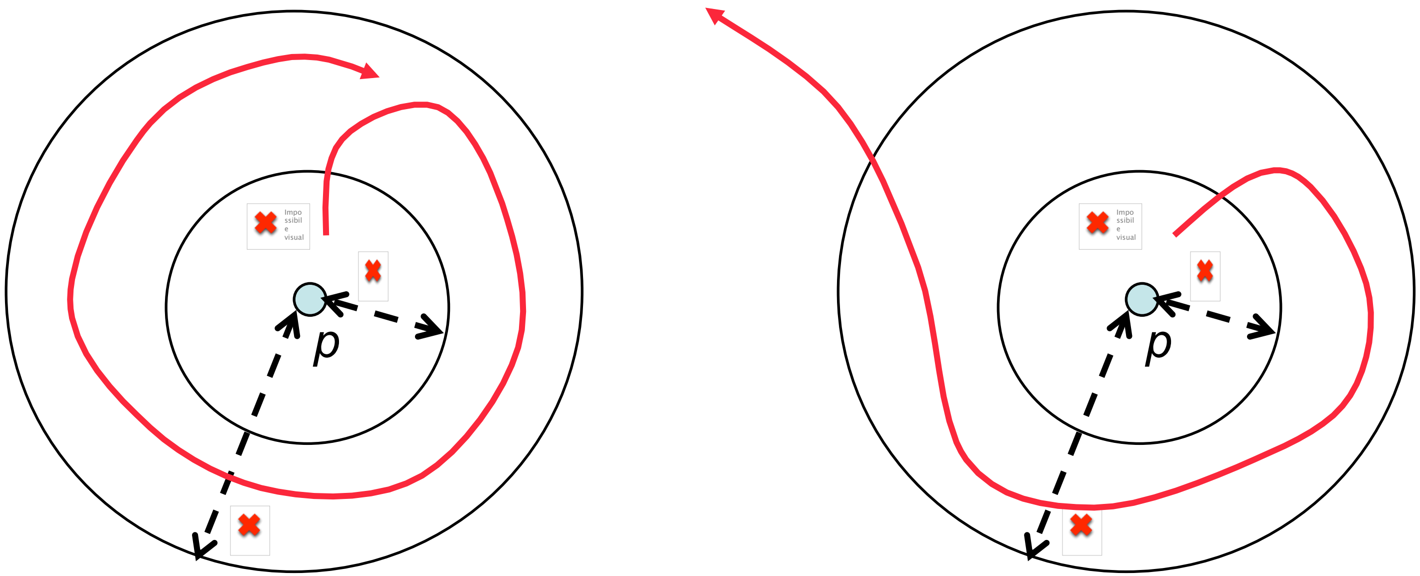
- A system is stable if with zero input, starting from any initial condition, the state trajectory converges to zero.
$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} e^{At}x(0) = 0.$$
- $\mathcal{L}(e^{At}) = (sI - A)^{-1}$. The polynomial $\det(sI - A)$ is called the characteristic polynomial.
- The system is stable if and only if all the roots of the characteristic polynomial have negative real part.
- Stability also implies that bounded input will produce bounded output.

In the following we focus on the following stability notions:

- Global Asymptotic Stability (GAS)
- Input-to-State Stability (ISS)
- Incremental Global Asymptotic Stability (δ-GAS)
- Incremental Input-to-State Stability (δ-ISS)

Global Asymptotic Stability

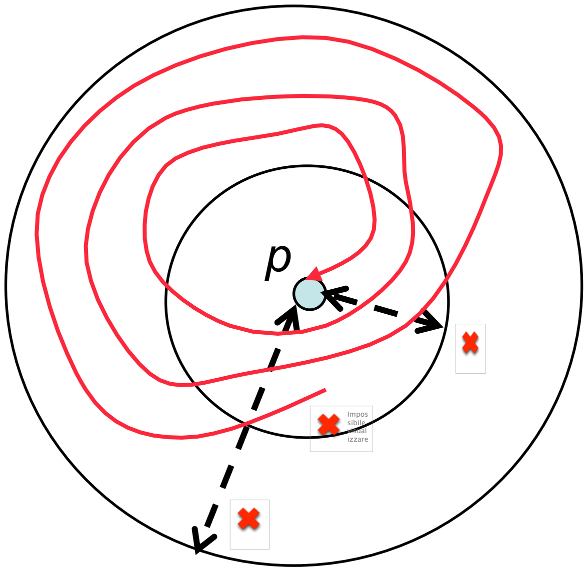
- Given $\dot{x} = f(x)$, let p be an equilibrium, i.e. $f(p) = 0$.
- The equilibrium p is stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p.



stable

Global Asymptotic Stability

- The equilibrium p is asymptotically stable if for any $\epsilon > 0$, there is a $\delta(\epsilon)$, such that the trajectory with initial condition x_0, with $\|x_0 - p\| < \delta(\epsilon)$ remains within ϵ distance from p and converge to p.



Asymptotically stable

1- A continuous function α: $[0,a) \rightarrow [0,\infty)$ is said to be a class K function if it is strictly increasing and $\alpha(0) = 0$. Function α: $[0,\infty) \rightarrow [0,\infty)$ is said to be a class K_∞ if it is a class K function and $\lim_{r\rightarrow\infty} \alpha(r) = \infty$.

2- A continuous function β: $[0,a) \times [0,\infty) \rightarrow [0,\infty)$ is said to be a class KL function if for each fixed s, function $\beta(r,s)$ is a class K function and for each fixed r, function $\beta(r,s)$ is decreasing and tends to zero as s goes to ∞.

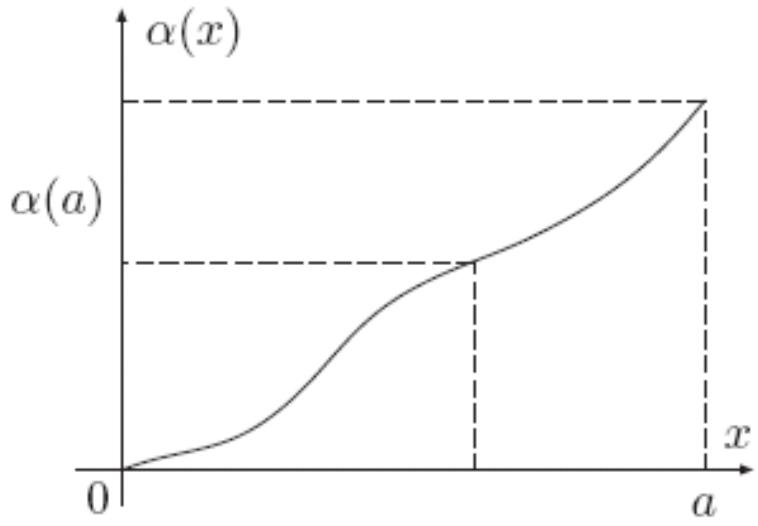


Figura 8 – Funzione di classe $\mathcal{K}$

Global Asymptotic Stability

The equilibrium x=0 of $\dot{x} = f(x,0)$ is Globally Asymptotically Stable (GAS) if there exists a KL function **β** so that for any $t \geq 0$, $y \in R^n$ and **u = 0**

$\| \mathbf{x}(t,y,0) \| \leq \beta(\|y\|, t)$

Theorem:

The equilibrium x=0 of $\dot{x} = f(x,0)$ is GAS if there exists a smooth function $V : R^n \rightarrow R^+$ such that:

i) $V(x) \geq 0$ for all $x \in R^n$ and $V(x) = 0$ if and only if $x = 0$

ii) $dV/dx\ f(x,0) < 0$ for all $x \in R^n$

Further details from H. K. Khalil, Nonlinear Systems, Prentice Hall, 1996

A control system $\dot{x} = f(x,u)$ is Input-to-State Stable (ISS) if there exist a KL function β and a K_∞ function γ so that for any $t \geq 0$, $y \in R^n$ and $\mathbf{u}$

$$\| \mathbf{x}(t,y,\mathbf{u}) \| \leq \beta(\|y\| , t) + \gamma(\|\mathbf{u}\|_\infty)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is ISS if there exists a smooth function $V : R^n \rightarrow R^+$ and K_∞ functions α_1, α_2, ρ, σ such that:

i) $\alpha_1 (\|x\|) \leq V(x) \leq \alpha_2 (\|x\|)$ for all $x \in R^n$

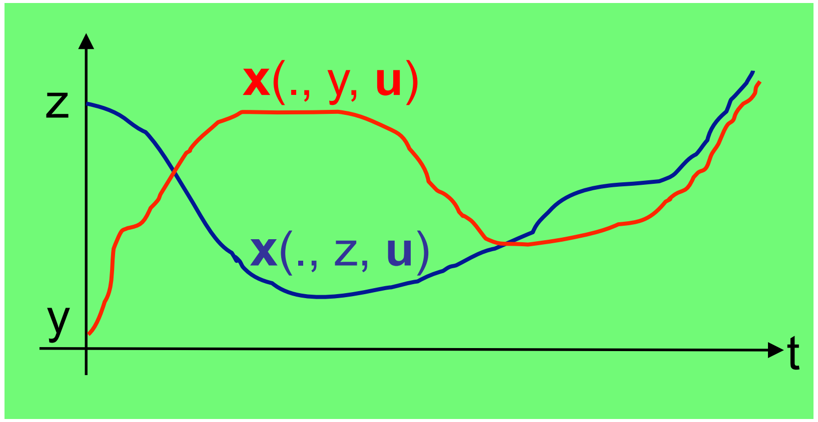
ii) $dV/dx\, f(x,\mathbf{u}) < - \rho (\|x\|) + \sigma (\|u\|)$ for all $x \in R^n$ and $u \in R^m$

Further details from H. K. Khalil, Nonlinear Systems, Prentice Hall, 1996

Incremental Global Asymptotic Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Globally Asymptotically Stable (δ-GAS) if there exists a KL function β so that for any $t \geq 0$, $y,z \in R^n$ and $\mathbf{u}$

$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) || \leq \beta(||y-z||, t)$$



Additional details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Global Asymptotic Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Globally Asymptotically Stable (δ-GAS) if there exists a KL function $\boldsymbol{\beta}$ so that for any $t \geq 0$, $y,z \in R^n$ and $\mathbf{u}$

$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{u}) || \leq \beta(||y-z||, t)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is δ-GAS if there exists a smooth function $V : R^n \times R^n \rightarrow R^+$ and K_∞ functions α_1, α_2, ρ such that:

i) $\alpha_1 (|| x - y ||) \leq V(x, y) \leq \alpha_2 (|| x - y ||)$ for all $x,y \in R^n$

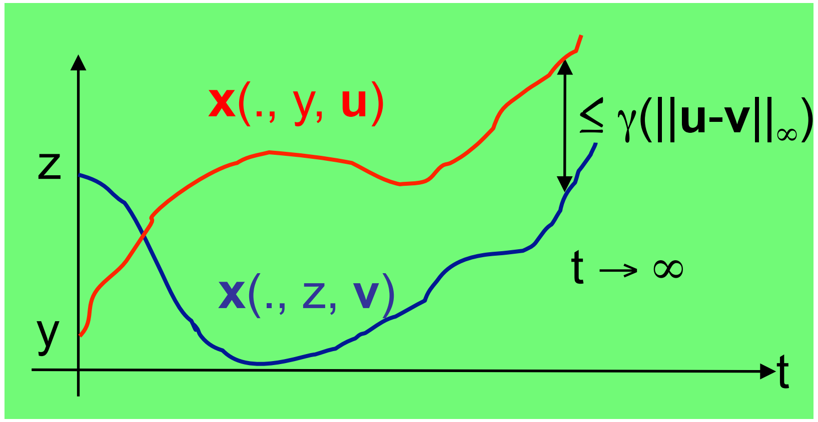
ii) $dV/dx\, f(x,u) + dV/dy\, f(y,u) < - \rho (|| x - y ||)$ for all $x,y \in R^n$ and $u \in R^m$

Additional details in D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Incremental Input-State Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Input-to-State Stable (δ-ISS) if there exist a KL function β and a K_∞ function γ so that for any $t \geq 0$, $y, z \in R^n$ and **u**, **v**

$$|| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) || \leq \beta(||y - z||, t) + \gamma(||\mathbf{u} - \mathbf{v}||_\infty)$$



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Incremental Input-State Stability

A control system $\dot{x} = f(x,u)$ is Incrementally Input-to-State Stable (δ-ISS) if there exist a KL function $\boldsymbol{\beta}$ and a K_∞ function γ so that for any $t \geq 0$, $y, z \in R^n$ and **u**, **v**

$$\| \mathbf{x}(t,y,\mathbf{u}) - \mathbf{x}(t,z,\mathbf{v}) \| \leq \beta(\|y - z\|, t) + \gamma(\|\mathbf{u} - \mathbf{v}\|_\infty)$$

Theorem:

A control system $\dot{x} = f(x,u)$ is δ-ISS if there exists a smooth function $V : R^n \times R^n \rightarrow R^+$ and K_∞ functions $\alpha_1, \alpha_2, \rho, \sigma$ such that:

i) $\alpha_1 (\|x - y\|) \leq V(x, y) \leq \alpha_2 (\|x - y\|)$ for all $x,y \in R^n$

ii) $dV/dx\, f(x,u) + dV/dy\, f(y,v) < - \rho (\|x - y\|) + \sigma (\|u - v\|)$ for all $x,y \in R^n$ and $u,v \in R^m$

Further details from D. Angeli, A Lyapunov approach to incremental stability properties, IEEE-TAC 02

Connections among Stability Notions

$$
\begin{tikzcd}
\delta\text{-ISS} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "f(0,0)=0"'] & \delta\text{-GAS} \arrow[d, Rightarrow, "f(0,0)=0"] \\
\text{ISS} \arrow[r, Rightarrow] & \text{GAS}
\end{tikzcd}
$$

Homework:

1) Prove such connections!
2) How do these notions specialize to the case of linear control systems?